Surface waves on water of non-uniform depth

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1. Introduction

Gravity waves occur on the surface of a liquid such as water, and the manner in which they propagate depends upon its depth. Although this dependence is described in principle by the equations of the 'exact linear theory' of surface waves, these equations have not been solved except in some special cases. Therefore, oceanographers have been unable to use the theory to describe surface wave propagation in water whose depth varies in a general way. Instead they have employed a simplified geometrical optics theory for this purpose (see, for example, Sverdrup & Munk (1944)). It has been used very successfully, and consequently various attempts, only partially successful, have been made to deduce it from the exact linear theory. It is the purpose of this article to present a derivation which appears to be satisfactory and which also yields corrections to the geometrical optics theory.

The present derivation is based upon an asymptotic solution of the equations of the exact linear theory for periodic waves in water of arbitrary non-uniform depth. The solution is asymptotic in the sense that the depth and wavelength must be small compared to the horizontal scale length of the bottom contour. Our solution is formal since we do not prove its asymptotic character. Therefore, it is significant that the first term agrees exactly with the asymptotic form of the solution for waves in water with a uniformly sloping bottom, as the bottom slope tends to zero. This asymptotic form was deduced by Friedrichs (1948). The accuracy of our result is indicated by the fact that for a bottom slope angle of 6°, the asymptotic form agrees very well with the exact solution at all points beyond one wavelength from the shore (see Stoker 1947).

Previously Lowell (1949) has shown that in shallow water the geometrical optics theory follows from the linear shallow-water theory. The latter is a simplified form of the exact linear theory which applies when the depth is small compared to the wavelength. The significance of this restriction is not clear, in view of the fact that Lowell's derivation depends upon letting the wavelength tend to zero. Our derivation seems to clarify this point since we require the depth to tend to zero in a fixed ratio to the wavelength, while the horizontal scale of the bottom contour is constant. We could have obtained the same results by keeping the depth and wavelength constant while permitting the horizontal scale length of the bottom contour to become infinite.

A previous attempt by the present author to derive the geometrical optics theory from the exact linear theory was based upon an asymptotic solution obtained by requiring the wavelength alone to tend to zero. The resulting geometrical optics theory was independent of the depth. That result can be explained by noting that waves disturb the water only within a few wavelengths of the surface. As the wavelength tends to zero, the wave motion is confined to a vanishingly thin layer near the surface, so the waves do not 'feel' the bottom. This derivation, which may be useful for other purposes, is contained in Appendix I.

We will now describe the geometrical optics theory to which we have already referred. In this theory a propagation velocity is defined at each point P of the surface. This velocity is just that which waves of the given period would have in water having uniform depth equal to that at P. By using this propagation velocity and employing Fermat's principle of optics, rays can be defined. It is assumed that surface waves propagate along these rays. The variation of the amplitude of these waves along a ray is then determined by using the principle of conservation of energy. In its optical form this principle states that the flux of energy is the same at all cross-sections of a tube of rays. In the present case a tube is bounded by two neighbouring rays. The energy flux is proportional to the square of the amplitude of the waves and to the distance between the rays. It is assumed that at each point P the proportionality factor, which depends upon the depth, is the same as for plane waves in water of uniform depth equal to the depth at P. From these assumptions the variations of amplitude along a ray can be determined. This amplitude variation has been studied by Miche (1944).

2. Formulation of the problem

We consider the periodic irrotational motion of an inviscid incompressible liquid bounded below by the rigid surface $Y = -H(x,z)$ and bounded above by an unknown free surface. In the absence of motion the free surface is the plane $Y=0$; when motion of angular frequency ω occurs, it is $Y = \mathscr{R}\{e^{i\omega t}\eta(x,z)\}$. Here t denotes the time, η is the complex amplitude of the surface wave motion, and the positive Y-axis points vertically upward. In the exact linear theory of surface waves, the wave amplitude η is given by

$$\eta(x,z) = (i\omega/g)\Phi(x,0,z). \tag{1}$$

Here $\Phi(x, Y, z)$ is the velocity potential of the fluid motion, and g is the acceleration of gravity.

The velocity potential satisfies the following conditions (see Stoker 1947)

$$\Delta\Phi = 0 \quad \text{in} \quad 0 \geqslant Y \geqslant -H(x,z), \tag{2}$$

$$\Phi_Y = \beta\Phi \quad \text{on} \quad Y=0 \qquad (\beta = \omega^2/g), \tag{3}$$

$$\Phi_Y + H_x\Phi_x + H_z\Phi_z = 0 \quad \text{on} \quad Y = -H(x,z). \tag{4}$$

Furthermore, Φ satisfies other conditions which we will consider later.

We now introduce the new quantities y, h and ϕ defined by

$$y = \beta Y, \qquad h = \beta H, \qquad \phi(x,y,z) = \Phi(x, Y, z). \tag{5}$$

In terms of these quantities, (2), (3) and (4) become

$$\beta^2\phi_{yy} + \phi_{xx} + \phi_{zz} = 0 \quad \text{in} \quad 0 \geqslant y \geqslant -h(x,z), \tag{6}$$

$$\phi_y = \phi \quad \text{on} \quad y = 0, \tag{7}$$

$$\beta^2\phi_y + h_x\phi_x + h_z\phi_z = 0 \quad \text{on} \quad y = -h(x,z). \tag{8}$$

The problem we consider is that of finding solutions ϕ of (6), (7) and (8) for large values of β.

As the first step towards finding solutions, we introduce three unknown functions $k(x,z)$, $S(x,z)$ and $A(x,y,z,\beta)$. Without loss of generality we may express ϕ in terms of these functions by the relation

$$\phi = A\cosh[k(y+h)]e^{i\beta S}. \tag{9}$$

Next we insert (9) into (6), (7) and (8). These equations then become the following, in which $\alpha \equiv k(y+h)$ and $\nabla \equiv (\partial/\partial x, \partial/\partial z)$:

$$\beta^2[\{k^2 - (\nabla S)^2\}A\cosh\alpha + A_{yy}\cosh\alpha + 2kA_y\sinh\alpha] + \\ + i\beta[(\nabla^2 S)A\cosh\alpha + 2\nabla S\,.\,\nabla(A\cosh\alpha)] + \nabla^2(A\cosh\alpha) = 0; \tag{10}$$

$$A_y\cosh kh + kA\sinh kh = A\cosh kh \qquad y = 0; \tag{11}$$

$$\beta^2 A_y + i\beta A\nabla h\,.\,\nabla S + \nabla h\,.\,\nabla A = 0 \qquad y = -h. \tag{12}$$

We now want solutions of (10), (11) and (12) for large β which represent waves travelling on the surface.

3. Solution of the problem

To solve (10), (11) and (12), we assume that, for large β, A has an asymptotic expansion of the form

$$A \sim A_0(x,z) + \sum_{n=1}^{\infty} A_n(x,y,z)/(i\beta)^n. \tag{13}$$

Then we insert (13) into (10), assuming that termwise differentiation is correct. When we equate coefficients of each power of β, we obtain the following equations:

$$(\nabla S)^2 = k^2; \tag{14}$$

$$(A_1)_{yy}\cosh\alpha + 2k(A_1)_y\sinh\alpha = 2\nabla S\,.\,\nabla(A_0\cosh\alpha) + A_0\cosh\alpha\nabla^2 S; \tag{15}$$

$$(A_n)_{yy}\cosh\alpha + 2k(A_n)_y\sinh\alpha = 2\nabla S\,.\,\nabla(A_{n-1}\cosh\alpha) + A_{n-1}\cosh\alpha\nabla^2 S + \\ + \nabla^2(A_{n-2}\cosh\alpha) \qquad (n \geqslant 2). \tag{16}$$

In a similar way, we obtain from (11) the equations

$$k\tanh kh = 1, \tag{17}$$

$$(A_n)_y = 0 \qquad y = 0 \qquad (n \geqslant 1). \tag{18}$$

Finally, from (12), we obtain

$$(A_1)_y = A_0\nabla h\,.\,\nabla S \qquad y = -h, \tag{19}$$

$$(A_n)_y = A_{n-1}\nabla h\,.\,\nabla S + \nabla h\,.\,\nabla A_{n-2} \qquad y = -h \qquad (n \geqslant 2). \tag{20}$$

Let us now examine the equations (14) to (20) which must be satisfied by k, S and the A_n. Equation (17) is a transcendental equation which determines $k(x,z)$ in terms of the known depth $h(x,z)$. This is the same equation that occurs in the case of uniform depth and its solution is well known. Once k is known, (14) is the eiconal equation of geometrical optics for S. If the value of S is prescribed at each point of any given curve, the solution S can be found by means of the characteristics of (14). These characteristics are just the optical rays in a medium of refractive index k, and the surfaces $S =$ const. are the corresponding wave-fronts. In terms of these rays the directional derivative $\Delta S \,.\, \nabla f$, which appears in (15) and (16) is just $k(df/d\tau)$ where τ measures arc-length along a ray.

Next let us consider the amplitudes A_n which satisfy (15) and (16). These equations can be simplified by means of the identity

$$(A_n)_{yy} \cosh \alpha + 2k(A_n)y \sinh \alpha = (\cosh \alpha)^{-1}[(A_n)_y \cosh^2 \alpha]_y. \tag{21}$$

Now (21) is inserted into (15), which becomes

$$[(A_1)_y \cosh^2 \alpha]_y = (2\nabla S \,.\, \nabla A_0 + A_0 \nabla^2 S)\cosh^2 \alpha + A_0 \nabla S \,.\, \nabla \cosh^2 \alpha. \tag{22}$$

Now we integrate (22) from zero to y, noting that, by (18), $(A_1)_y = 0$ at $y = 0$. Thus (22) yields

$$(A_1)_y \cosh^2 \alpha = (2\nabla S \,.\, \nabla A_0 + A_0 \nabla^2 S + A_0 \nabla S \,.\, \nabla) \int_0^y \cosh^2 \alpha \, dy. \tag{23}$$

The integral in (23) is given by

$$\int_0^y \cosh^2 k(y+h)\, dy = \tfrac{1}{2} k^{-1} \times$$
$$\times [\sinh k(y+h) \cosh k(y+h) - \sinh kh \cosh kh] + \tfrac{1}{2} y. \tag{24}$$

We now use (24) in (23) and obtain

$$(A_1)_y \cosh^2 \alpha = \tfrac{1}{2}(2\nabla S \,.\, \nabla A_0 + A_0 \nabla^2 S + A_0 \nabla S \,.\, \nabla) \times$$
$$\times \{k^{-1}[\sinh \alpha \cosh \alpha - \sinh kh \cosh kh] + y\}. \tag{25}$$

From (25) we can calculate A_1 up to an additive function of x and z, once A_0 is known. The method which we will now use to find A_0 will also determine this additive function. For that purpose the equation (16) for $n = 2$ would have to be examined.

To find A_0 we set $y = -h$ in (25). Then we can eliminate $(A_1)_y$ by means of (19). In this way we obtain the following equation for A_0:

$$2A_0 \nabla h \,.\, \nabla S = -(2\nabla S \,.\, \nabla A_0 + A_0 \nabla^2 S)(\sinh^2 kh + h) + A_0 \nabla S \,.\, (\nabla h - \nabla \sinh^2 kh). \tag{26}$$

This can be written as

$$\nabla S \,.\, \nabla [A_0^2(\sinh^2 kh + h)] + [A_0^2(\sinh^2 kh + h)]\nabla^2 S = 0. \tag{27}$$

Since $\nabla S \,.\, \nabla = k(d/d\tau)$, (27) is an ordinary differential equation along a ray. Its solution is

$$A_0^2(\sinh^2 kh + h) = [A_0^2(\sinh^2 kh + h)]_{\tau_0} \exp\left\{ -\int_{\tau_0}^{\tau} k^{-1} \nabla^2 S \, d\tau \right\}. \tag{28}$$

Luneberg (1944) has shown that the exponential in (28) is given by

$$\exp\left\{-\int_{\tau_0}^{\tau} k^{-1}\nabla^2 S\, d\tau\right\} = \frac{k(\tau_0)}{k(\tau)}\frac{d\sigma(\tau_0)}{d\sigma(\tau)}. \qquad (29)$$

In (19), $d\sigma(\tau_0)/d\sigma(\tau)$ is the ratio of the width of a narrow strip of rays at τ_0 to its width at τ, or rather the limit of this ratio as the width tends to zero. When (29) is used in (28), the equation for the variation of A_0 along a ray finally becomes

$$A_0^2(\sinh^2 kh + h)k\, d\sigma = \text{const.} \qquad (30)$$

Equation (30) has a simple physical interpretation. Except for a constant factor, the left side of (30) is the energy flux per length $d\sigma$ of a plane wave in water of uniform depth h. Thus (30) expresses the fact that the energy flux is constant along a tube of rays.

4. Conclusion

Let us now summarize our results. The wave amplitude η is given by (1), the potential is given by (9), k is given by (17), S is determined by the eiconal equation (14), and A_0 is determined by (30). This yields the leading term in the solution for large β. It conforms with all the principles of the geometrical optics theory and thus provides a derivation of that theory. Additional terms are given by the A_n, which satisfy (16) and the boundary conditions (18) and (20).

In Appendix II our result is compared with the asymptotic form of the exact solution for plane waves normally incident on a uniformly sloping beach. That asymptotic form is obtained from the work of Friedrichs (1948). It agrees exactly with our result.

Our result is not valid at caustics of the ray system since $d\sigma$ is zero on them, and then (30) shows that A_0 is infinite. A method for obtaining the correct finite result at and near caustics was given by Kay & Keller (1954). At a shore line where $h = 0$, our result also fails because it yields an infinite value for A_0. In this case the solution for the uniformly sloping beach given by Peters (1952) can be used near the shoreline to yield the correct result there. Both of the exceptions just described are instances of 'boundary layers' in which the asymptotic behaviour of the solution is different from that outside these layers.

Appendix I. Another asymptotic solution

We will try to solve (2), (3) and (4) for Φ by writing

$$\Phi = ue^{i\beta S}. \qquad (\text{A}\,1)$$

In terms of u and S, (2) and (3) become

$$\beta^2(\nabla S)^2 u - i\beta[2\nabla S\,.\,\nabla u + u\nabla^2 S] + \nabla^2 u = 0, \qquad (\text{A}\,2)$$

$$i\beta S_y u + u_y = \beta u \qquad y = 0. \qquad (\text{A}\,3)$$

Here ∇ is the three-dimensional gradient and $y = Y$ is used for notational convenience. We now assume that u has the asymptotic expansion

$$u \sim \sum_{j=1}^{\infty} u_j(x,y,z)/(i\beta)^j. \tag{A 4}$$

Upon inserting (A 4) into (A 2) and equating coefficients of β, we obtain

$$(\nabla S)^2 = 0, \tag{A 5}$$

$$2\nabla S \,.\, \nabla u_j + u_j \nabla^2 S = -\nabla^2 u_{j-1} \qquad j \geqslant 0, \quad u_{-1} \equiv 0. \tag{A 6}$$

Similarly from (A 3) we obtain

$$iS_y = 1 \qquad y = 0, \tag{A 7}$$

$$(u_j)_y = 0 \qquad y = 0, \qquad j \geqslant 0. \tag{A 8}$$

From (A 5) we conclude that S is complex, so we write

$$S = R + iI; \tag{A 9}$$

where R and I are real. Now (A 5) yields the two equations

$$\nabla R \,.\, \nabla I = 0, \qquad (\nabla R)^2 = (\nabla I)^2. \qquad \text{(A 10), (A 11)}$$

Equation (A 10) shows that the wave-fronts $R =$ const. are orthogonal to the surfaces $I =$ const., which are essentially surfaces of constant amplitude. This is common in evanescent waves.

When (A 9) is used in (A 7), there results

$$R_y = 0, \qquad I_y = -1, \qquad y = 0. \tag{A 12}$$

To satisfy (A 10) and (A 12), we assume

$$I = -y, \qquad R = R(x,z). \qquad \text{(A 13), (A 14)}$$

Then (A 11) yields the eiconal equation for R

$$(\nabla R)^2 = 1. \tag{A 15}$$

To solve (A 6) with $j = 0$ for u_0, we assume that u_0 is real. Then (A 6) implies that $u_0 = u_0(x,z)$, and that

$$2\nabla R \,.\, \nabla u_0 + u_0 \nabla^2 R = 0. \tag{A 16}$$

As before, (A 16) is an ordinary differential equation along a ray. Its solution is

$$u_0(\tau) = u_0(\tau_0)[d\sigma(\tau_0)/d\sigma(\tau)]^{1/2}. \tag{A 17}$$

From (A 15) it follows that the rays are straight lines, and along any ray $R = R(\tau_0) + \tau$. Then $d\sigma(\tau_0)/d\sigma(\tau) = (\rho + \tau_0)/(\rho + \tau)$, where ρ denotes the radius of curvature of the wave-front at τ_0.

Upon collecting our results, we find

$$\Phi \sim A\tau^{-1/2} e^{i\beta\tau + \beta y}. \tag{A 18}$$

In writing (A 18) we have collected the factors independent of τ into the factor A and shifted the origin of τ so that $\rho + \tau$ becomes τ. In other words, we now measure τ from the centre of curvature of the wave-front—i.e. from the caustic—on each ray. The result (A 18) does indeed conform to the principles of geometrical optics, but the index of refraction to which it corresponds is unity. The waves travel with the velocity $\omega/\beta = g/\omega$, which is the velocity of waves of frequency ω in water of infinite depth. This was explained in the Introduction.

In constructing the solution (A 18) the condition (4) at the bottom was ignored and the result in (A 18) does not satisfy it. It is also impossible to satisfy this condition by including additional terms of the sum (A 4). Therefore, the present expansion applies in water of infinite depth, when condition (4) is absent. When the depth is finite we must modify the expression (A 1) by including another term of the same form, which we may think of as arising by reflection of the original term from the bottom. This reflected term must be so chosen that the two terms together satisfy condition (4) at the bottom. But then the reflected term violates the condition (3) at the top, which the first term alone satisfies. We therefore add another term, reflected from the top, to remedy this, then another reflected from the bottom, etc.

On the basis of these considerations, we assume that Φ has the following asymptotic expansion

$$\Phi \sim \sum_{n=0}^{\infty} \exp(i\beta S_n)\beta^{[n/2]} \sum_{j=0}^{\infty} u_{nj}/(i\beta)^j. \tag{A 19}$$

Here $[n/2]$ denotes the greatest integer not exceeding $n/2$, and the S_n and u_n are functions of x, y and z. The terms with $n = 0$ are those considered before. By proceeding as before we can obtain equations for all the new terms. It appears that at $y = 0$ all the S_n with $n \geqslant 1$ will have positive real parts which depend upon the depth. Therefore they will all be of exponentially lower order in β than the terms with $n = 0$ which we considered before. Therefore, our previous result will still determine the surface shape or wave pattern.

Appendix II. Comparison with an exact solution

Friedrichs (1948) has obtained the asymptotic form of the exact solution for waves on a sloping beach as the slope angle tends to zero. He considered only points on the top surface $y = 0$. However, it can be seen that his equation (16) also holds for $y < 0$ provided that x is replaced by $x+iy$ in (12). If we then treat y as small compared to x, we obtain

$$\lambda(x+iy) = \lambda(x) + i\omega y\lambda'(x) + \dots,$$

and

$$k[\lambda(x+iy)] = k[\lambda(x)] + i\omega y/\lambda(x) + \dots.$$

When these results are inserted into his (16) and a typographical error in the second exponent is corrected, (16) becomes

$$X_2^P(x,y,t) = -(1-\lambda)A(\lambda)\exp[i(\omega^{-1}k+t+\tfrac{1}{4}\pi)-y/\lambda] - \\ -(1+\lambda)A(\lambda)\exp[-i(\omega^{-1}k+t+\tfrac{1}{4}\pi)+y/\lambda] + \dots.$$

Then the real potential function ϕ is given by

$$\begin{aligned}\phi = \mathscr{R}\{X_2^P\} &= -A(\lambda)[(1-\lambda)e^{-y/\lambda} + (1+\lambda)e^{y/\lambda}]\cos(\omega^{-1}k+t+\tfrac{1}{4}\pi) \\ &= -2A(\lambda)[\cosh(y/\lambda) + \lambda\sinh(y/\lambda)]\cos(\omega^{-1}k+t+\tfrac{1}{4}\pi).\end{aligned}$$

Now we reintroduce dimensional variables by replacing x by βx, t by σt (σ is the angular frequency and $\beta = \sigma^2/g$) and we define the depth $h = \beta\omega x$ (ω is the slope of the bottom). We also introduce $k' = \lambda^{-1}$. Then (12) becomes $k' \tanh k'h = 1$. Now ϕ is given by

$$\phi = -2A(\lambda)\frac{\cosh[k'(y+h)]}{\cosh k'h}\cos(\omega^{-1}k+\sigma t+\tfrac{1}{4}\pi).$$

From the definition of $A(\lambda)$, it follows that $A(\lambda) = A_0 \cosh k'h$, where A_0 is given by our equation (30). Thus, ϕ finally becomes

$$\phi = -2A_0\cosh[k'(y+h)]\cos(\omega^{-1}k+\sigma t+\tfrac{1}{4}\pi).$$

This is exactly the sum of two expressions of the form (9) given by our asymptotic solution.

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